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STATICALLY INDETERMINATE FRAMED STRUCTURES OF NON-LINEAR ELASTIC MATERIAL

F. E. ARCHER

School of Civil Engineering, University of New South Wales,
Kensington, N.S.W., Australia

Abstract—In this paper a general method of analysing framed structures composed on non-linear elastic material is described. For statically indeterminate pin-jointed frames composed of material having a stress–strain relation of the form $\varepsilon = \varepsilon_0(\sigma/\sigma_0)^n$ and for rigid jointed frames composed of members having moment–curvature relations of the form $\phi = \phi_0(M/M_0)^n$ it is shown that no redistribution of actions occurs as the loading is increased. The applicability of the method of "mixed systems" is demonstrated. For moment–curvature relationships having a definite slope at $M = 0$ it is shown in a number of particular cases that redistribution under increasing load is only slight and that the distribution of actions is similar to that obtained by a linear analysis.

1. INTRODUCTION

NUMEROUS methods of analysis of statically indeterminate framed structures of linear elastic material have been developed during the past half century. These methods apply only to materials which behave in a Hookean manner. On the other hand, few, if any, of the materials used in real structures are linear-elastic. Mild steel is very nearly linear-elastic up to the yield point, but beyond the yield point is plastic, and use has been made of this plastic property in the "collapse" method of design of steel structures. Reinforced concrete, however, is not linear-elastic at any stage. In spite of this fact building authorities suggest that the internal actions in indeterminate frames of reinforced concrete are to be determined from a linear-elastic analysis.

In view of the fact that most structural materials are non-linear it is of interest to examine a method for analysing indeterminate structures of such materials and to note the effects on redistribution of certain forms of stress–strain relation.

Hoff [1] has shown that it is possible by means of an elastic analogue to eliminate the parameter time from the analysis of the final state of stress in bodies subject to creep. As, however, the stress–strain relations to be considered are in general non-linear, attention is again focussed on non-linear elastic analysis. As an example Hoff [1] has used the complementary energy method to determine the moment distribution in a frame for which the rate of change of curvature of each element was assumed as proportional to a power of the bending moment.

2. INDETERMINACY

The statement that a structure is statically indeterminate implies that the conditions of static equilibrium alone are insufficient to permit the determination of the internal actions in the structure.*

3. EQUILIBRIUM

Any framed structure which is m-fold statically indeterminate may be converted to a statically determinate structure by the introduction of m suitable releases. The releases cannot be chosen indiscriminately but must in fact be so chosen that all internal actions in the structure may be determined from the conditions of static equilibrium. In general, however, it will be found that there are several ways in which the releases may be made.

Corresponding to the m releases made to reduce the structure to a statically determinate state it can be shown that there are m independent systems of self-equilibrating member actions. If the structure is reduced to a one-fold indeterminate structure by introducing $(m-1)$ suitable releases and then unit value is postulated for the action corresponding to the mth release, it is possible to find, by the equations of statics, actions in members throughout the structure corresponding to this unit action. Such a system of forces or actions is referred to as a self-equilibrating system in that it is not related to any load system external to the structure. By repeating the process m times with a different and independent one-fold indeterminate structure each time, a total of m independent systems of self-equilibrating forces is obtained. Any number of other systems of self-equilibrating forces can be obtained by taking linear combinations of the above systems.

If $(P \,.\, f_{0i})$ is the generalized internal action (stress-resultant) at some section i of the structure, such that this system of actions is in equilibrium with the applied external loads described by P, and f_{ri} is the internal action at the same position, corresponding to the rth system of self-equilibrating forces, then the sum of the actions

$$Pf_{0i} + p_r \,.\, f_{ri}$$

will represent the internal action at position i in the structure such that the complete system is in equilibrium with the external loads. In the above p_r is an arbitrary multiplier. Any number of self-equilibrating systems with arbitrary multipliers together with the system Pf_{0i} will give a system of actions in equilibrium with the external loads. However, if it is decided to include the influence of actions at all the release positions (and this will be necessary if the complete m fold indeterminate structure is to be analyzed) it will be necessary to use all the m systems of self-equilibrating forces in the general expression for the internal action at i, i.e.

$$F_i = Pf_{0i} + \sum_{r=1}^{m} p_r \,.\, f_{ri}. \tag{1}$$

The terms Pf_{0i} in equation (1) can be derived by analysing the structure which has been made statically determinate, by the introduction of the above-mentioned m releases. However, it is not necessary that these actions be determined in this way. They may be derived by analysing the structure which has been made determinate by the introduction of releases other than those previously referred to. Nevertheless, any system of actions

* Henderson and Bickley [2] have proposed a systematic procedure by means of which the degree of static indeterminacy may be calculated. It is not intended to deal with this aspect of the problem here and the interested reader may consult the paper referred to for a discussion of this problem.

$(Pf_{0i})'$ corresponding to a particular choice of releases m will be found to be a combination of the actions Pf_{0i} and the terms f_{ri} with special values for the multipliers.

As an illustration of the above consider the structure shown in Fig. 1.

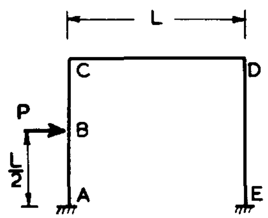

FIG. 1. Fixed-ended portal frame.

One possible set of releases are moment releases at A, C and D. Confining attention to bending moments the actions Pf_{0i}, f_{1i}, f_{2i} and f_{3i} are shown diagrammatically in Fig. 2.

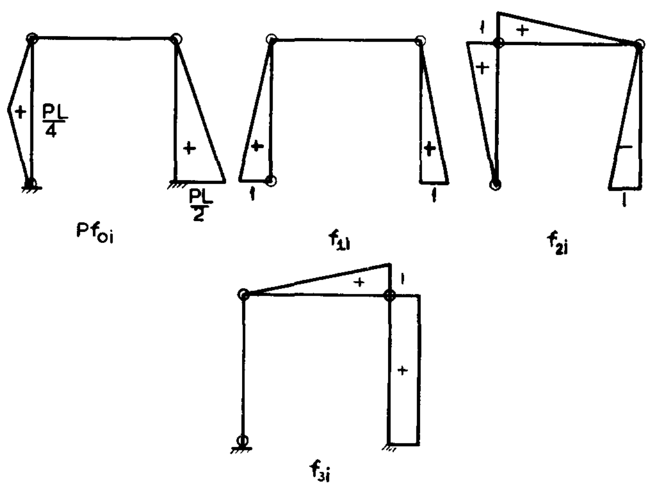

FIG. 2. Bending moment diagrams for the external load system, and self-equilibrating systems with hinges at A, C and D.

If one adds to Pf_{0i} the quantity $(-PL/2)(f_{1i})$ the new system $(Pf_{0i})' = Pf_{0i}-(PL/2)(f_{1i})$ is obtained. This system is shown diagrammatically in Fig. 3 and corresponds to a determinate structure obtained by making three releases at E.

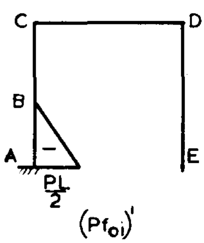

FIG. 3. Bending moment diagram for the external load system with three releases at E.

The complete specification of the internal actions by

$$F_i = Pf_{0i}+p_1f_{1i}+p_2f_{2i}+p_3f_{3i}$$

is identical with the specification

$$F_i = (Pf_{0i})'+p'_1f_{1i}+p_2f_{2i}+p_3f_{3i}$$

provided

$$p'_1 = [p_1+(PL/2)].$$

It can be shown further that the Pf_{0i} system need not necessarily refer to a statically determinate structure but may refer to an indeterminate one obtained from the original structure.

4. COMPATIBILITY

Although it was stated with reference to equation (1) that the multipliers p_r could have any values whatsoever, it must be remembered that equilibrium only was being considered. For arbitrary values of p_r it is possible that continuity will be destroyed at some or all of the release positions (e.g. axial separations, lateral displacements and/or rotations may occur). In order that compatibility be satisfied, that is that continuity be maintained throughout the structure, it will be necessary to determine particular values for the multipliers p_r. This operation will involve the calculation of deformations of the elements of the structure, and these will depend upon the dimensions of members and upon the stress–strain relation of the material of which the members are composed.

5. PIN-JOINTED FRAMED STRUCTURES

In a structure composed of straight members joined together through frictionless ball-joints the actions through the structure are axial forces only, provided the external loads are applied at node points. Consider such a structure with 'm' redundants and for simplicity let each member be of uniform cross-section throughout its length. Assume further that the stress–strain relation for the material is $\varepsilon = \varepsilon_0 \,.\, G(\sigma/\sigma_0)$, where σ_0 is a limiting stress and ε_0 is the corresponding strain.

The axial force in the ith member, F_i, may be expressed as explained earlier as

$$F_i = Pf_{0i}+\sum_{r=1}^{m} p_r \,.\, f_{ri}\,.$$

The stress in the member will be given by

$$\sigma_i = F_i/A_i$$

and, making use of the stress–strain relation, the strain in the member will be

$$\varepsilon_i = \varepsilon_0 \,.\, G(F_i/\sigma_0 A_i).$$

The elongation of this member will be

$$e_i = \varepsilon_i \,.\, l_i$$

$$= \varepsilon_0 l_i G\left(\frac{F_i}{\sigma_0 A_i}\right).$$

The principle of virtual work may be stated in the following way. If any system of forces is in equilibrium, the work done by the forces during any virtual displacements is equal to zero.

Each of the self-equilibrating force systems referred to earlier may be taken with any pattern of virtual displacements and according to the principle of virtual work the work done by such a force system will be zero. If the virtual displacements are chosen as those in the indeterminate structure acted upon by the external loads P, and remembering that there are no discontinuities in this structure, then the total work will be given as the product of the force f_{ri} and the elongation e_i for each member making up the self-equilibrating force system structure. Hence

$$\begin{aligned} W_T &= \sum f_{ri} \cdot e_i \\ &= \sum \varepsilon_0 l_i \cdot f_{ri} \cdot G\left(\frac{F_i}{\sigma_0 A_i}\right) \\ &= 0 \end{aligned}$$

the summation being taken over all members making up the self equilibrating force system structure.

Using the expression for F_i gives;

$$\sum \varepsilon_0 l_i \cdot f_{ri} \cdot G\left[\frac{Pf_{0i}+p_1 f_{1i}+-\ldots+p_m f_{mi}}{\sigma_0 A_i}\right] = 0 \tag{2}$$

Repeating the procedure for m independent self-equilibrating systems will yield m simultaneous equations in which the unknowns are $p_1 \ldots p_m$.

Except for the particular case where

$$\varepsilon = \text{const.}\, \sigma$$

the simultaneous equations are not linear. Further, in general, $p_1 \ldots p_m$ will be non-linear functions of the load term P.

If the stress–strain relation for unloading is different from that for loading of a material then the method which is summarized in the system of equations (2) is not valid if unloading occurs in any part of the structure during the general loading process.

6. SPECIAL STRESS–STRAIN RELATIONSHIP

If the stress–strain relation for the material has the special form

$$\varepsilon = \varepsilon_0 \left(\frac{\sigma}{\sigma_0}\right)^n$$

where n is any number greater than one, but not necessarily an integer, then equation (2) becomes

$$\sum \varepsilon_0 l_i \cdot f_{ri}\left[\frac{Pf_{0i}+p_1 f_{1i}+\ldots p_m f_{mi}}{\sigma_0 A_i}\right]^n = 0.$$

This can be re-written in the form

$$\frac{\varepsilon_0}{\sigma_0^n} P^n \sum l_i f_{ri}\left[\frac{f_{0i}+(p_1/P) f_{1i}+\ldots (p_m/P) \cdot f_{mi}}{A_i}\right]^n = 0.$$

As $(\varepsilon_0 P^n/\sigma_0^n)$ is non-zero we can divide throughout by that quantity. In the resulting system of non-linear simultaneous equations the unknowns are

$$(p_1/P), (p_2/P), \ldots (p_m/P).$$

For such a stress–strain relation, then, the ratio of forces in the various members will be independent of the intensity of the load system characterized by P, provided that the structure is subjected to proportional loading. In this limited sense the principle of superposition applies in this special case. Further, as the relative distribution of forces in the members remains constant as the external loads are increased, no unloading occurs in any member and the method of analysis is therefore valid.

Example 1

Consider the symmetrical three bar structure of Fig. 4. The bars are of equal cross-sectional area A and the material of each bar has a stress–strain relation $\varepsilon = \beta\sigma^n \ldots n > 1$

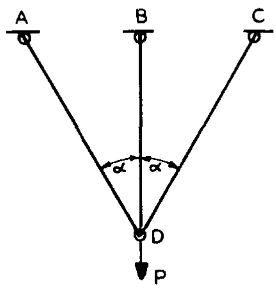

FIG. 4. Symmetrical three-bar system.

Let

$$P_{AD} = P_{CD} = P_1$$

and

$$P_{BD} = P_2.$$

To satisfy equilibrium

$$P = P_2 + 2P_1 \cos\alpha.$$

The stress in AD and CD is P_1/A and the corresponding strain is $\beta(P_1/A)^n$. The elongation of member AD is $\beta(P_1/A)^n . L \sec\alpha$. The vertical displacement of D corresponding to this elongation is

$$\beta\left(\frac{P_1}{A}\right)^n L \sec^2\alpha.$$

The stress in member BD is P_2/A, and the corresponding strain is $\beta(P_2/A)^n$.

The elongation of member BD and consequently the vertical displacement of D is,

$$\beta\left(\frac{P_2}{A}\right)^n . L.$$

To satisfy the compatibility condition

$$\beta\left(\frac{P_1}{A}\right)^n L\sec^2\alpha = \beta\left(\frac{P_2}{A}\right)^n . L$$

from which

$$P_1 . \sec^{2/n}\alpha = P_2.$$

Substitution in the equilibrium equation yields

$$P_1 = \frac{P(\cos\alpha)^{2/n}}{1+2(\cos\alpha)^{1+2/n}}$$

and so

$$P_2 = \frac{P}{1+2(\cos\alpha)^{1+2/n}}.$$

Table 1 shows the variation in P_2/P for various values of n for the particular case $\alpha = 60°$.

TABLE 1

n	P_2/P
1	0·800
2	0·666
3	0·615
4	0·586
5	0·569
∞	0·500

It is of interest to note that as n becomes very large the result approaches the result obtained by assuming that the material of the bars is "rigid-plastic". If the rigid-plastic approach is made there is no unique solution so long as the force in all members is below the force to cause plastic flow—any distribution of forces satisfying the equilibrium conditions will be valid. If, however, the material has a stress–strain relation

$$\varepsilon = \varepsilon_0\left(\frac{\sigma}{\sigma_0}\right)^n$$

then for the situation dealt with here the distribution of forces in the bars is unchanged from the beginning of loading up to any value of the applied load.

Example 2

Consider the unsymmetrical frame of Fig. 5. The bars in this structure are all of cross-sectional area A, and the stress–strain relation is

$$\varepsilon = \varepsilon_0\left(\frac{\sigma}{\sigma_0}\right)^n$$

Using the notation of the earlier section, the member forces and sizes are shown in Table 2.

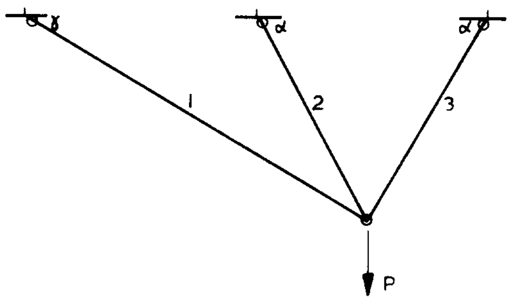

FIG. 5. Unsymmetrical three-bar system.

TABLE 2

Member	Pf_{0i}	f_{1i}	$F_2 = Pf_{0i}+p_1f_{1i}$
1	0	1	p_1
2	$\frac{P}{2}\operatorname{cosec}\alpha$	$-\frac{1}{2}\left(\frac{\cos\gamma}{\cos\alpha}+\frac{\sin\gamma}{\sin\alpha}\right)$	$\frac{P}{2}\operatorname{cosec}\alpha-\frac{p_1}{2}\left(\frac{\cos\gamma}{\cos\alpha}+\frac{\sin\gamma}{\sin\alpha}\right)$
3	$\frac{P}{2}\operatorname{cosec}\alpha$	$-\frac{1}{2}\left(\frac{\sin\gamma}{\sin\alpha}-\frac{\cos\gamma}{\cos\alpha}\right)$	$\frac{P}{2}\operatorname{cosec}\alpha-\frac{p_1}{2}\left(\frac{\sin\gamma}{\sin\alpha}-\frac{\cos\gamma}{\cos\alpha}\right)$

As a particular case take $\alpha = 60°$ and $\gamma = 30°$, then the work equation

$$\sum l_i f_{1i}\cdot\left(\frac{F_i}{A}\right)^n = 0$$

becomes

$$-2\left[1-2\left(\frac{p_1}{P}\right)\right]^n+\left[1+\left(\frac{p_1}{P}\right)\right]^n+(3)^{1+n/2}\cdot\left(\frac{p_1}{P}\right)^n = 0.$$

Expanding the above gives

$$-1+5\,.\,{}^nC_1\,.\left(\frac{p_1}{P}\right)-7\,.\,{}^nC_2\left(\frac{p_1}{P}\right)^2+17\,.\,{}^nC_3\left(\frac{p_1}{P}\right)^3\ldots-(2^n-1)\,.\,{}^nC_{n-1}\left(\frac{p_1}{P}\right)^{n-1}$$
$$+\{2^{n+1}+1+(3)^{1+n/2}\}\left(\frac{p_1}{P}\right)^n = 0$$

if n is an odd integer, and

$$-1+5\,.\,{}^nC_1\left(\frac{p_1}{P}\right)-7\,.\,{}^nC_2\left(\frac{p_1}{P}\right)^2+17\,.\,{}^nC_3\left(\frac{p_1}{P}\right)^3\ldots+(2^n+1)\,.\,{}^nC_{n-1}\left(\frac{p_1}{P}\right)^{n-1}$$
$$-\{2^{n+1}-1-(3)^{1+n/2}\}\left(\frac{p_1}{P}\right)^n = 0$$

if n is an even integer.

For any value of n it is seen that p_1 is independent of the value of P. In this particular problem there is no difficulty associated with taking n either odd or even as the forces in the members are all positive for all values of P and the corresponding elongations obtained in terms of F_i^n are all of the same sign.*

* If the loading or form of the structure were such that the force in some members was *negative* the use of an even value for n would make the changes in length of those members positive instead of negative.

The values of the forces in the three members are shown in Table 3 for several values of n.

TABLE 3

n	P_1/P	P_2/P	P_3/P
1	0·09815	(0·80370) $\sqrt{3}/3$	(1·09815) $\sqrt{3}/3$
2	0·09807	(0·80386) $\sqrt{3}/3$	(1·09807) $\sqrt{3}/3$
3	0·0734	(0·8532) $\sqrt{3}/3$	(1·0734) $\sqrt{3}/3$
3·5	0·0635	(0·8730) $\sqrt{3}/3$	(1·0635) $\sqrt{3}/3$
4	0·0560	(0·8880) $\sqrt{3}/3$	(1·0560) $\sqrt{3}/3$
5	0·0453	(0·9094) $\sqrt{3}/3$	(1·0453) $\sqrt{3}/3$

As n increases P_1/P decreases while P_2/P and P_3/P approach the value $\sqrt{3}/3$. For this particular structure the rigid-plastic analysis would give $P_1/P = 0$, and $P_2/P = P_3/P = \sqrt{3}/3$.

Example 3

The structure considered in Examples 1 and 2 were one-fold indeterminate only. The next example to be considered will be the two-fold indeterminate structure of Fig. 6.

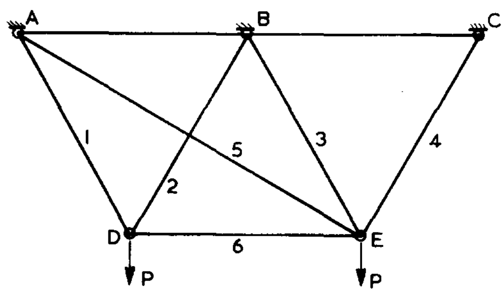

FIG. 6. Two-fold indeterminate plane pin-jointed frame.

All bars have the same cross-sectional area which may be taken as unity and all bars except 5 are of length L. The forces in the bars are given in Table 4 in terms of the external load and unit values of forces in members 5 and 6.

TABLE 4

Member	Length	Pf_{0i}	f_{5i}	f_{6i}	F_i
1	L	$\sqrt{3}/3P$	0	1	$\sqrt{3}/3P+p_6$
2	L	$\sqrt{3}/3P$	0	-1	$\sqrt{3}/3P-p_6$
3	L	$\sqrt{3}/3P$	$-2\sqrt{3}/3$	-1	$\sqrt{3}/3P-2\sqrt{3}/3p_5-p_6$
4	L	$\sqrt{3}/3P$	$\sqrt{3}/3$	1	$\sqrt{3}/3P+\sqrt{3}/3p_5+p_6$
5	$\sqrt{3}L$	0	1	0	p_5
6	L	0	0	1	p_6

If it is assumed that the stress–strain relation for the material is given by

$$\varepsilon = \varepsilon_0(\sigma/\sigma_0)^n$$

then the parameters p_5 and p_6 will be obtained by solving the equations

$$\sum_{i=1}^{6} (F_i)^n \,.\, f_{5i} \,.\, l_i = 0$$

$$\sum_{i=1}^{6} (F_i)^n \,.\, f_{6i} \,.\, l_i = 0.$$

For the case $n = 1$, that is for a linear elastic material the above equations become,

$$(5+3\sqrt{3})(p_5/P)+3\sqrt{3}\,(p_6/P)-1 = 0$$

and

$$\sqrt{3}\,(p_5/P)+5(p_6/P) = 0$$

the solution of which is

$$p_5/P = 0{\cdot}1191,$$

$$p_6/P = -0{\cdot}0412.$$

The forces in the members of the frame are then $P_1 = 0{\cdot}536P$, $P_2 = 0{\cdot}619P$, $P_3 = 0{\cdot}481P$, $P_4 = 0{\cdot}605P$, $P_5 = 0{\cdot}119P$ and $P_6 = -0{\cdot}041P$. For the case $n = 3$, that is $\varepsilon = \varepsilon_0\sigma^3/\sigma_0^3$ the equations become with $q_5 = p_5/P$ and $q_6 = p_6/P$,

$$(17+9\sqrt{3})q_5^3+27\sqrt{3}\,q_5^2q_6+45q_5q_6^2-21q_5^2-18\sqrt{3}\,q_5q_6+15q_5$$
$$+9\sqrt{3}\,q_6^3-9q_6^2+9\sqrt{3}\,q_6-1 = 0$$

and

$$\sqrt{3}\,q_5^3+5q_5^2q_6+3\sqrt{3}\,q_5q_6^2-\sqrt{3}\,q_5^2-2q_5q_6+\sqrt{3}\,q_5+5q_6^3+4\,q_6 = 0$$

that is a pair of simultaneous equations involving powers of q_5 and q_6. These equations have been solved by a semi-graphical method. Curves were drawn to represent the relation between q_5 and q_6 given by each of the equations, points on the curves being obtained by assuming values for q_6 and then solving the resulting cubic equations in q_5. The final solution was obtained as the co-ordinates of the point of intersection of the two curves. The results obtained in this way were $p_5/P = 0{\cdot}130$ and $p_6/P = -0{\cdot}053$. The corresponding bar forces in the structure are $P_1 = 0{\cdot}524P$, $P_2 = 0{\cdot}630P$, $P_3 = 0{\cdot}480P$, $P_4 = 0{\cdot}599P$, $P_5 = 0{\cdot}130P$ and $P_6 = -0{\cdot}053P$.

As a check on the conditions of compatibility the extensions and contractions of the bars were calculated as proportional to $(F_i/A_i)^3 \,.\, l_i$ and a Williot displacement diagram drawn. This is reproduced in Fig. 7. The displacement diagram confirms that compatibility is satisfied.

7. STRUCTURES CONTAINING FLEXURAL MEMBERS ONLY

A great number of statically indeterminate structures, including continuous beams and rigid jointed plane frames where the length of the members are large compared with the cross-sectional dimensions, are able to sustain loads applied to them mainly by virtue

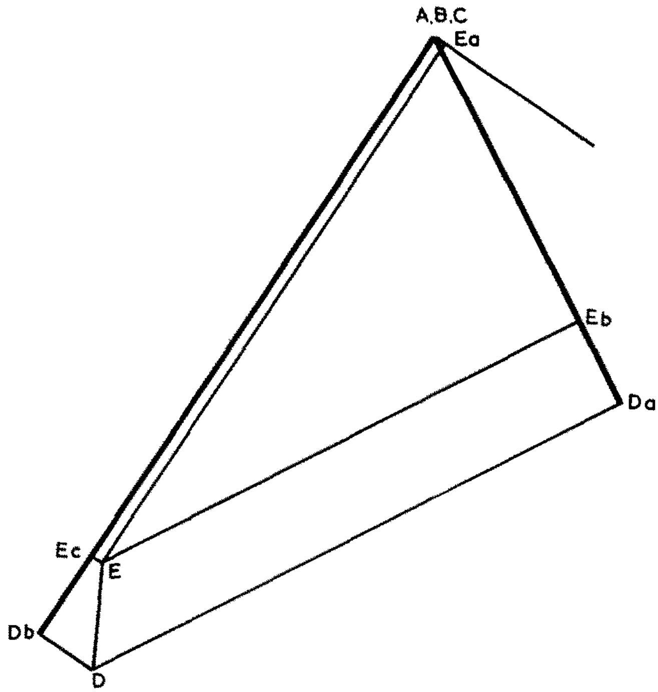

FIG. 7. Williot displacement diagram for the frame shown in Fig. 6.

of the bending resistance of the members. The deformations which occur in these structures are predominantly flexural deformations. Because of the relatively insignificant deformations of other kinds, i.e. axial and shear, it is reasonable to ignore them in the analysis of structures in this class.

The statements made in the earlier section dealing with equilibrium apply to all framed structures. In this section, as flexural deformations only are to be considered, the internal actions which concern us are the bending moments throughout the structure and equation (1) may be used as the general expression for the bending moment at section i in the structure. As we are concerned at this stage with bending moment only it may be less confusing if the general expression for bending moment is written as

$$M_i = Pm_{0i} + \sum_{r=1}^{m} p_r m_{ri}$$

where the terms have meanings similar to those given for the terms in equation (1).

Instead of using a stress–strain relation it is convenient to deal with moment–curvature relationships, which of course depend upon both the member cross-sectional dimensions and on the stress–strain relation for the material. Thus it can be assumed that the moment–curvature relation is of the form

$$\phi_i = G(M_i)$$

Application of the principle of virtual work gives

$$W_T = \int_A \phi_i \,.\, m_{ri}\, \mathrm{d}s = 0\,;$$

where $\int_A$ represents integration over the complete structure. Expanding this equation gives

$$\int_A m_{ri} \,.\, G(Pm_{0i} + \sum_{r=1}^{m} p_r \,.\, m_{ri}) \,.\, \mathrm{d}s = 0.$$

By taking in turn the m values $m_{1i} - m_{mi}$ there results m simultaneous equations in the unknown $p_1 \ldots p_m$.

8. SPECIAL MOMENT–CURVATURE RELATION

If the moment–curvature relation is given by

$$\phi = \phi_0 \,.\, (M/M_0)^n \,*$$

then the 'm' work equations are of the form

$$\int_A m_{ri} \,.\, \phi_0 \left[\frac{Pm_{0i} + \sum_{r=1}^{m} p_r m_{ri}}{M_0}\right]^n \mathrm{d}s = 0$$

which can be rewritten as

$$\frac{\phi_0 P^n}{M_0^n} \int_A m_{ri} \,.\, \left(m_{0i} + \sum_{r=1}^{m} \frac{p_r}{P} m_{ri}\right)^n \mathrm{d}s = 0.$$

It is permissible to divide throughout by $\phi_0(P/M_0)^n$. In so doing it is seen that the resulting simultaneous equations have (p_r/P) as the unknowns. In other words, for structures with members having moment–curvature relations of the form given above, no redistribution of moments occurs as the external loads are increased, provided of course the loading is "proportional". In this limited sense then the principle of superposition may be applied in such cases.

We now consider some examples.

The fixed-end beam

The fixed-end beam of uniform section with uniformly distributed load is one-fold indeterminate. Such a beam together with the two moment diagrams m_{0i} and m_{1i} is shown in Fig. 8.

The bending moment at any point is given by

$$M_i = \frac{wL}{2} \,.\, x - \frac{wx^2}{2} - p_1$$

which, using the substitutions

$$k = x/L \quad \text{and} \quad q = wL^2/2$$

* M_0 is some limiting value of bending moment and ϕ_0 is the corresponding curvature.

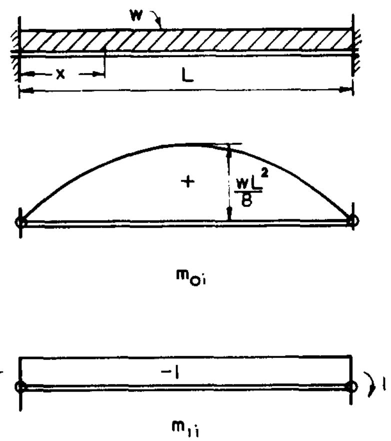

FIG. 8. Fixed-ended beam with uniformly distributed load, and bending moment diagrams corresponding to moment releases at the ends of the beam.

becomes

$$M_i = q(k-k^2) - p_1.$$

Consider the case for which the moment–curvature relation for the beam is given by

$$\phi/\phi_0 = (M/M_0)^n$$

n being an odd positive integer.

Then the equation for the determination of p_1 is

$$\int_0^1 (-1)(M_i)^n \, dk = 0$$

i.e.,

$$\int_0^1 [q(k-k^2)-p_1]^n \, dk = 0$$

or with

$$S = p_1/q; \qquad \int_0^1 [(k-k^2)-S]^n \, dk = 0.$$

Using the Binomial Expansion and then integrating we obtain the following equation;

$$-S^n + {}^nC_1\left[\frac{1}{2}-\frac{1}{3}\right]S^{n-1} - {}^nC_2\left[\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right]S^{n-2} + {}^nC_3\left[\frac{1}{4}-\frac{3}{5}+\frac{3}{6}-\frac{1}{7}\right]S^{n-3}$$

$$-{}^nC_4\left[\frac{1}{5}-\frac{{}^4C_1}{6}+\frac{{}^4C_2}{7}-\frac{{}^4C_3}{8}+\frac{{}^4C_4}{9}\right]S^{n-4} - \ldots + \left[\frac{1}{n+1}-\frac{{}^nC_1}{n+2}-\ldots-\frac{1}{2n+1}\right] = 0.$$

For $n = 1$, that is for a linear elastic material,

$$-S+\tfrac{1}{6} = 0$$

or

$$p_1 = wL^2/12 = 0{\cdot}0833wL^2.$$

For $n = 3$, the equation becomes, $140S^3-70S^2+14S-1 = 0$; which has a solution, $S = 0{\cdot}151$ and hence $p_1 = 0{\cdot}0750wL^2$. For $n = 5$, the equation becomes $2772S^5-2310S^4+924S^3-198S^2+22S-1 = 0$, which has a solution, $S = 0{\cdot}1380$, and hence $p_1 = 0{\cdot}0690wL^2$. For $n = 7$, the equation is $51480S^7-60{,}060S^6+36036S^5-12870S^4+2860S^3-390S^2+30S-1 = 0$ from which $S = 0{\cdot}1315$ and hence $p_1 = 0{\cdot}0657wL^2$.

In the foregoing n was chosen as an odd integer. This was done so that the sign of the curvature would correspond with the sign of the bending moment at all points along the beam. If n were chosen as an even integer the sign of the curvature would be unchanged as the bending moment changed sign.

By changing the approach slightly the case of n even can be dealt with.

As the general expression for M is $M = q(k-k^2)-p$, the point of zero moment, k_0 say is found by solving the quadratic

$$k_0-k_0^2-(p_1/q) = 0$$

or $2k_0 = 1-\sqrt{(1-4S)}$; where $S = p_1/q$.

It is seen that for $k < k_0$ the bending moment is negative but for $k_0 < k < \frac{1}{2}$ the bending moment is positive.

Taking account of this sign change the equation which must be solved for S becomes

$$\int_0^{k_0}(-1)(M_i)^n\,dk-\int_{k_0}^{\frac{1}{2}}(-1)(M_i)^n\,dk = 0$$

or

$$\int_0^{k_0}[(k-k^2)-S]^n\,dk-\int_{k_0}^{\frac{1}{2}}[(k-k^2)-S]^n\,dk = 0.$$

For the particular case $n = 2$, the above equation becomes,

$$(24k_0^5-60k_0^4+40k_0^3-1)-S(-80k_0^3+120k_0^2-10)+S^2(120k_0-30) = 0;$$

but $S = k_0-k_0^2$. Substituting for S yields the equation

$$64k_0^5-130k_0^4+100k_0^3-40k_0^2+10k_0-1 = 0;$$

from which $k_0 = 0{\cdot}194$, and hence $S = 0{\cdot}1564$ and $p_1 = 0{\cdot}0782wL^2$. The variation of p_1 with n is shown in Table 5.

TABLE 5

n	1	2	3	5	7
p_1/wL^2	0·0833	0·0782	0·0750	0·0690	0·0657

In this problem p_1wL^2 is the value of the fixed-end moment.

It is interesting to note that as n increases the value of p_1 approaches 0·0625 which is the value for the "rigid-plastic" solution. That this result might be expected is seen by examining Fig. 9 in which moment–curvature curves of the form

$$\phi/\phi_0 = (M/M_0)^n$$

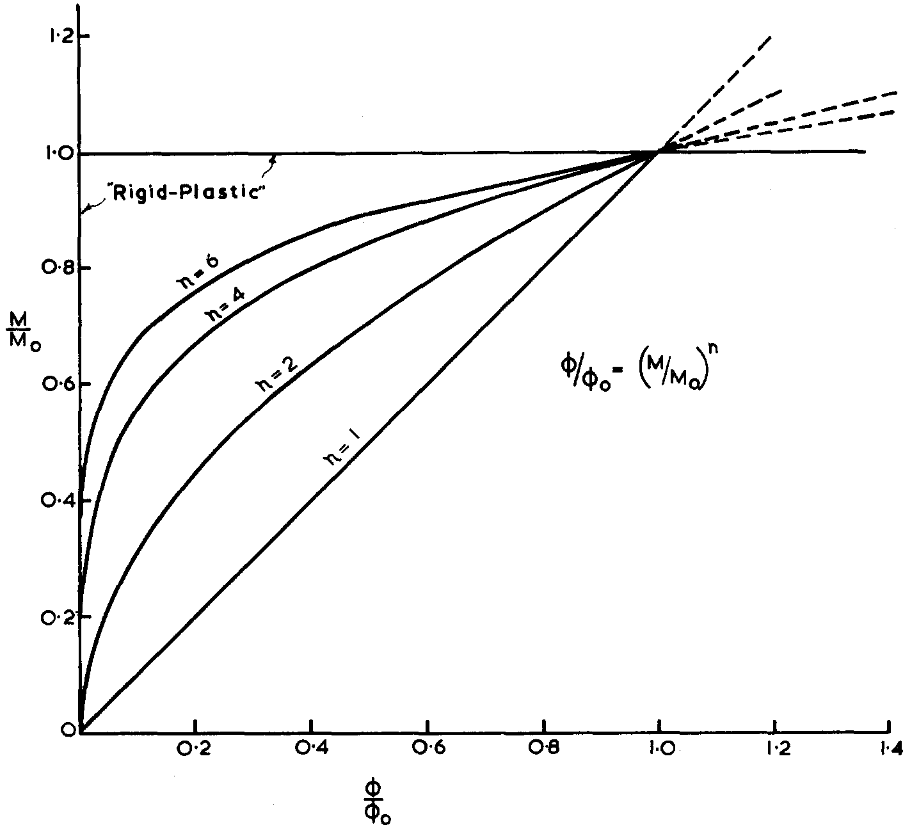

FIG. 9. Moment–curvature diagrams of the form $\phi/\phi_0 = (M/M_0)^n$.

are drawn for various values of n. In the same figure the moment–curvature curve for the rigid-plastic case is also shown.

It is noted again that for the case curvature proportional to a power of moment there is no relative redistribution of moments throughout the beam as the loading is increased. Such moment–curvature relation of course has certain peculiarities; as M approaches zero, $dM/d\phi \to \infty$ and further $dM/d\phi \neq 0$ for any value of M.

The example dealt with in this section has been one-fold indeterminate only. To indicate the greater difficulty, from the numerical point of view, associated with structures having more than one indeterminate two additional examples are given.

The fixed-end beam of uniform section loaded with a concentrated load P is shown in Fig. 10, together with the bending moment diagrams representing Pm_{0i}, m_{1i} and m_{2i}.

The general expression for M_i is

$$M_i = Px + p_1 + p_2 x/L - \cdots (-L/4 < x < 0)$$

and

$$M_i = p_1 + p_2 x/L \qquad (0 < x < 3L/4).$$

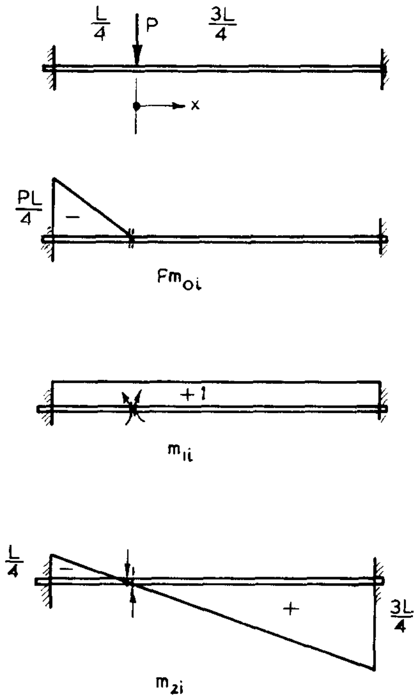

FIG. 10. Fixed-ended beam with a concentrated load and bending moment diagrams corresponding to moment and shear releases at the point load.

The virtual work equations in this case become

$$\int_{-L/4}^{0} (M_i)^n \, dx + \int_{0}^{3L/4} (M_i)^n \, dx = 0$$

and

$$\int_{-L/4}^{0} x(M_i)^n \, dx + \int_{0}^{3L/4} x(M_i)^n \, dx = 0.$$

If we choose $n = 3$, after expanding and integrating the above equations become

$$1024q_1^3 + 768q_1^2q_2 + 448q_1q_2^2 + 80q_2^3 - 96q_1^2 + 32q_1q_2 - 3q_2^2 + 16q_1 - 3q_2 - 1 = 0$$

and

$$1280q_1^3 + 2240q_1^2q_2 + 1200q_1q_2^2 + 244q_2^3 + 80q_1^2 - 30q_1q_2 + 3q_2^2 - 15q_1 + 3q_2 + 1 = 0.$$

where

$$q_1 = p_1/PL \quad \text{and} \quad q_2 = p_2/PL.$$

It is seen that in this case it is necessary to solve a pair of simultaneous equations involving powers of q_1 and q_2. Using a semi-graphical method together with Newton's method for finding root of a polynomial equation the following results were obtained:

$$p_1 = 0{\cdot}085PL$$

$$p_2 = -0{\cdot}205PL.$$

With these values for p_1 and p_2 the final bending-moment diagram is as shown in Fig. 11.

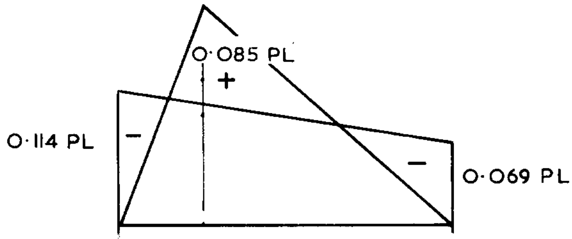

FIG. 11. Final bending moment diagram for the beam of Fig. 10.

9. MIXED SYSTEMS

Works [3], [4] on the linear analysis of framed structures have drawn attention to the fact that it is possible to make use of different release systems for the external load actions and the self-equilibrating systems. It may not be obvious that mixed systems can also be employed in the analysis of frames having non-linear moment-curvature relations.

Referring to the earlier section of this paper dealing with "Equilibrium" it is seen that the internal actions in the members of a frame may be expressed as

$$F_i = Pf_{0i} + \sum_{r=1}^{m} p_r f_{ri}$$

and the Pf_{0i} term, corresponding to one release system, may be changed to (Pf_{0i}) corresponding to some other release system merely by adding to Pf_{0i} some or all of the f_{ri} terms with suitable multipliers. As the deformations are some function of the internal actions, the deformations also may be represented by terms which have been derived from different release systems.

As an illustration of mixed systems consider the rigid frame shown in Fig. 12.

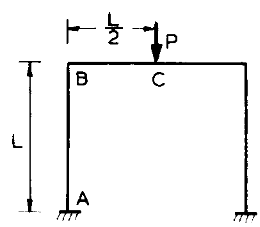

FIG. 12. Fixed-ended portal frame with symmetrical loading.

As the structure is symmetrical and is loaded symmetrically it is two-fold indeterminate. Assume that the moment curvature relation is

$$\phi/\phi_0 = (M/M_0)^3$$

(i) In the first instance we will choose three releases at C for the external load system and the self-equilibrating systems. The corresponding moment diagrams are shown in Fig. 13.

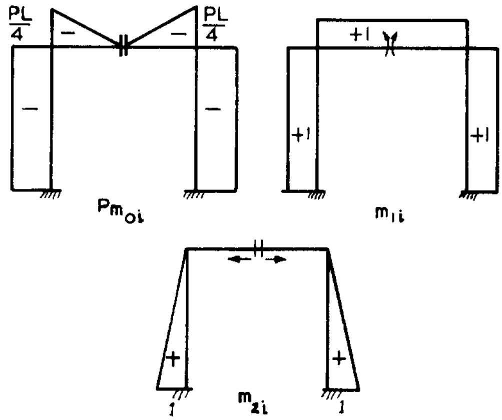

FIG. 13. Bending moment diagrams for the frame of Fig. 12 with three releases at C.

With this representation of the moments, the work equations involving q_1 and q_2; ($q_1 = p_1/PL$, $q_2 = p_2/PL$) are

$$768q_1^3+768q_1^2q_2+512q_1q_2^2+128q_2^3-480q_1^2-384q_1q_2-128q_2^2+112q_1+48q_2-9 = 0.$$

and

$$320q_1^3+640q_1^2q_2+480q_1q_2^2+128q_2^3-240q_1^2-320q_1q_2-120q_2^2+60q_1+40q_2-5 = 0.$$

Solving these equations gives

$$p_1 = 0{\cdot}145PL \quad \text{and} \quad p_2 = 0{\cdot}168PL.$$

The final bending moments in the structure are then

$$M_A = PL[-0{\cdot}250+0{\cdot}145+0{\cdot}168] = 0{\cdot}063PL$$

$$M_B = PL[-0{\cdot}250+0{\cdot}145] = -0{\cdot}105PL$$

$$M_C = PL[0{\cdot}145] = 0{\cdot}145PL.$$

(ii) As an example of the use of mixed systems now choose the $P \, . \, m_{0i}$ system shown in Fig. 14, together with the m_{1i} and m_{2i} of the previous case.

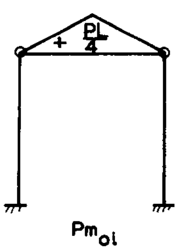

FIG. 14. Bending moment diagram for the frame of Fig. 12 for the external load system and moment releases at the ends of the beam.

The work equations are now

$$512q_1^3+768q_1^2q_2+512q_1q_2^2+128q_2^3+256q_1^3+96q_1^2+16q_1+1 = 0$$

and

$$10q_1^3+20q_1^2q_2+15q_1q_2^2+4q_2^3 = 0.$$

where $q_1 = p_1/PL$ and $q_2 = p_2/PL$. From the second equation

$$q_1/q_2 = -0{\cdot}623.$$

Substituting in the first equation and solving the resulting cubic equation gives $q_2 = 0{\cdot}168$, from which $q_1 = -0{\cdot}105$ or $p_1 = -0{\cdot}105PL$, and $p_2 = 0{\cdot}168PL$.

The final bending moments in the structure are

$$M_A = PL[0-0{\cdot}105+0{\cdot}168] = 0{\cdot}063PL$$

$$M_B = PL[0-0{\cdot}105+0] = -0{\cdot}105PL$$

$$M_C = PL[0{\cdot}250-0{\cdot}105+0] = 0{\cdot}145PL$$

which are the same as those previously obtained.

10. OTHER MOMENT–CURVATURE RELATIONSHIPS

As it has been shown that there is no redistribution of bending moments in structures having the special moment–curvature relationship $-\phi/\phi_0 = (M/M_0)^n$ it is of interest to examine the effect on redistribution of moments of other forms of moment–curvature relations.

Consider the beam of Fig. 8, but now let the moment–curvature relation be given by $\phi/\phi_0 = 1-\sqrt{[1-(M/M_0)]}$ for $\phi < \phi_0$ and $|M| < |M_0|$. For this curve $dM/d\phi$ has a definite value at $M = 0$ and for $M = M_0$, $dM/d\phi = 0$. In order that the moment and curvature should always be of the same sign in the beam considered, the expression is rewritten as

$$\phi/\phi_0 = [1-\sqrt{\{[1-p_1/M_0]+qk/M_0-qk^2/M_0\}}]$$

when the total moment is $-$ve, i.e., for $0 < k < k_0$, and

$$\phi/\phi_0 = [1-\sqrt{\{[1+p_1/M_0]-qk/M_0+qk^2/M_0\}}]$$

when the total moment is $+$ve, i.e. for $k_0 < k < \frac{1}{2}$. As before $q = wL^2/2$.

The work equation from which p_1 may be determined is

$$\int_0^{k_0} \phi \, dk+\int_{k_0}^{\frac{1}{2}} \phi \, dk = 0.$$

As before k_0 is given by

$$2k_0 = 1-\sqrt{[1-4p_1/q]}.$$

After performing the integrations we obtain,

$$-4\{1-\sqrt{[1-4p_1/q]}\}+2\sqrt{[1-p_1/M_0]}$$
$$-\sqrt{(q/M_0)}\left(\{1+4(M_0-p_1)/q\}\left\{\sin^{-1}\frac{\sqrt{[1-4p_1/q]}}{\sqrt{[1+4(M_0-p_1)/q]}}-\sin^{-1}\frac{1}{\sqrt{[1+4(M_0-p_1)/q]}}\right\}\right.$$
$$\left.+\{4(M_0+p_1)/q-1\}\ln\frac{\sqrt{[4(M_0+p_1)/q-1]}}{\sqrt{[4M_0/q-\sqrt{[1-4p_1/q]}}}\right)=0.$$

It is seen that p_1 is not independent of the ratio of q and M_0. If $wL^2/8 = M_0$, and we let $Y = p_1/M_0$ the equation for p_1 is

$$-2+3\sqrt{(1-Y)}-((2-Y)\{\sin^{-1}\sqrt{[(1-Y)/(2-Y)]}-\sin^{-1}\sqrt{[1/(2-Y)]}\}$$
$$+Y\ln(\sqrt{Y})/[1-\sqrt{(1-Y)}])=0.$$

The solution to this equation is $Y = 0{\cdot}655$, i.e., $p_1 = 0{\cdot}655wL^2/8 = 0{\cdot}0819wL^2$. For the case $wL^2/12 = M_0$, with $Y = p_1/M_0$ the equation for p_1 is

$$-4+4\sqrt{[(3-2Y)/3]}+2\sqrt{(1-Y)}$$
$$-\sqrt{6}\left([(5-2Y)/3]\left\{\sin^{-1}\sqrt{\frac{(3-2Y)/3}{(5-2Y)/3}}-\sin^{-1}\frac{1}{\sqrt{[(5-2Y)/3]}}\right\}\right.$$
$$\left.+[(2Y-1)/3]\ln\frac{\sqrt{[(2Y-1)/3]}}{\sqrt{(2/3)}-\sqrt{[(3-2Y)/3]}}\right)=0.$$

The solution to this equation is $Y = 0{\cdot}975$, i.e., $p_1 = 0{\cdot}975wL^2/12 = 0{\cdot}0812wL^2$. If $wL^2/4 = M_0$, then

$$-4+4\sqrt{(1-2Y)}+2\sqrt{(1-Y)}-\sqrt{2}\left((3-2Y)\left\{\sin^{-1}\sqrt{\frac{1-2Y}{3-2Y}}-\sin^{-1}\sqrt{\frac{1}{3-2Y}}\right\}\right.$$
$$\left.+(1+2Y)\ln\frac{\sqrt{(1+2Y)}}{\sqrt{2}-\sqrt{(1-2Y)}}\right)=0.$$

The solution in this case is $Y = 0{\cdot}331$ and hence $p_1 = 0{\cdot}331wL^2/4 = 0{\cdot}0827wL^2$. The variation of p_1 with wL^2/M_0 is shown in Table 6.

TABLE 6

wL^2/M_0	p_1/wL^2
4	0·0827
8	0·0819
12	0·0812

For the linear elastic case $p_1 = 0{\cdot}0833wL^2$. From the above table it is seen that some redistribution of moments occurs as the load intensity is increased from $4M_0/L^2$ to $12M_0/L^2$, however, the greatest departure from the linear elastic solution is only 2·5 per cent.

The above analysis is to a certain extent approximate only. As a slight redistribution of moment is indicated as the loading is increased, the position of the point of zero moment changes, i.e. k_0 decreases slightly. This means that over a very small length of beam sections which were subjected to small negative moments are subjected positive moments as the loading increases; or in other words a small amount of unloading occurs. The moment changes involved, however, are very small and occur about the point of zero moment. At this position on the moment–curvature curve the difference in slope of the tangent and the chord is insignificant and the effect of this unloading therefore may be neglected.

As a further example of a moment–curvature relation having a definite slope at $M = 0$ and zero slope at $M = M_0$ consider the case

$$(\pi/2)(\phi/\phi_0) = \sin^{-1}(M/M_0);$$

applied to the case of the fixed-end beam with symmetrically placed point loads P as shown in Fig. 15.

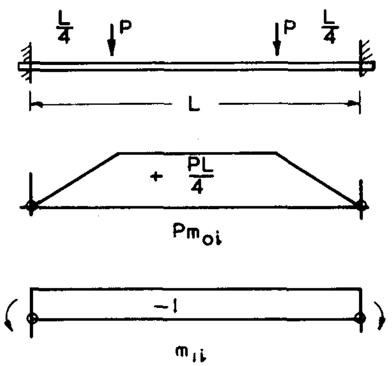

FIG. 15. Fixed-ended beam with symmetrical loading and bending moment diagrams corresponding to moment releases at the ends of the beam.

The work equation for the determination of p_1 in this case becomes

$$\int_0^{L/4} \sin^{-1}[(Px/M_0)-(p_1/M_0)]\,dx + \int_{L/4}^{L/2} \sin^{-1}[(PL/4M_0)-(p_1/M_0)]\,dx = 0$$

After integration the equation becomes

$$\begin{aligned}(4M_0/PL)\{[(PL/4M_0)-(p_1/M_0)]\sin^{-1}[(PL/4M_0)-(p_1/M_0)]\\ +\sqrt{\{1-[(PL/4M_0)-(p_1/M_0)]^2\}}-(p_1/M_0)\sin^{-1}(p_1/M_0)\\ -\sqrt{[1-(p_1/M_0)^2]}\}+\sin^{-1}[(PL/4M_0)-(p_1/M_0)] = 0\end{aligned}$$

This equation has been solved for various values of PL/M_0 and the results are shown in Table 7.

TABLE 7

PL/M_0	p_1/PL
2	0·187
4	0·186
5	0·185

Compared with these results p_1/PL would be 0·1875 for the linear elastic case; whereas the rigid-plastic analysis would give 0·125.

As in the previous example there is a slight redistribution of moments as the load increases, but for all levels of the load considered the distribution of moments does not differ from that for the linear elastic case by more than 1·3 per cent.

If we examine the behaviour of the fixed end beam of Fig. 15 and assume that the moment-curvature relation is given by $A(\phi/\phi_0) = (M/M_0)+B[(M/M_0)^3]$ so that the curve has a definite slope at $M = 0$, but has a positive slope for all other values of M the following results are obtained (see Table 8).

TABLE 8

A	B	PL/M_0	p_1/PL
2	1	2	0·185
2	1	4	0·181
2	1	5	0·179
3	2	2	0·184
3	2	4	0·178
3	2	5	0·175
5	4	2	0·181
5	4	4	0·175
5	4	5	0·171

For any of these three moment–curvature relations the amount of redistribution with increasing load is very slight, and the greatest divergence of the fixed-end moment from that obtained for a linear analysis is only 8·5 per cent.

CONCLUSIONS

1. In the case of pin-jointed frameworks which are statically indeterminate and which are composed of materials having a stress–strain relation of the form $\varepsilon/\varepsilon_0 = (\sigma/\sigma_0)^n$, where n is any positive number, no redistribution of member forces occurs as the external loads on the structure are increased, provided that, if more than one concentrated load is applied, the loading is increased proportionally. In this limited sense the principle of superposition of loads is valid.

If the value of n is made large it is found that the distribution of member forces approaches the distribution obtained by plastic analysis.

2. For plane rigid jointed indeterminate structures, in which axial and shear deformations may be ignored, and which are composed of members having moment–curvature relations of the form $\phi/\phi_0 = (M/M_0)^n$ no redistribution of moments occurs as the external loads are increased.

As for pin-jointed frameworks the "rigid-plastic" solution is approximated by making n large.

3. If the method of analysis employed is one in which the primary unknowns are "actions", then "mixed" systems may be employed irrespective of the form of the stress–strain relation of the materials.

4. It has been shown in a limited number of examples of structures with symmetrical

load systems and with non-linear moment–curvature relationships other than the power type, that although redistribution of moments does occur with increasing loads this redistribution is only slight and the distribution of moments is only slightly different from that obtained by linear analysis. It is not possible, however, to make generalizations regarding the behaviour of such structures with unsymmetrical loading.

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Résumé—Dans cette étude une méthode générale pour analyser les structures enchassées composées de materiel elastique non-linéaire est décrite. Pour les chassis assemblés statiquement indeterminés, composés de materiel ayant des relations de résistance contrainte de la forme $\varepsilon = \varepsilon_0(\sigma/\sigma_0)^n$ et pour des chassis assemblés rigides, composés de pièces ayant des relations de moment–courbine de la forme $\phi = \phi_0(M/M_0)^n$, il est indiqué qu'aucune redistribution d'action ne survient lorsque le chargement est augmenté. L applicabilité de la méthode de "systèmes mixtes" est démontrée. Pour des relations de moment–courbure ayant une pente définie à $M = 0$, il est démontré dans un nombres de cas particuliers que la redistribution sous un chargement augmentant n'est que légère et que la distribution d'actions est similaire à celle obtenue par l'analyse linéaire.

Zusammenfassung—Eine allgemeine Methode für die Berechnung von Rahmen, zusammengesetzt aus nichtlinearem elastischen Material, ist in dieser Abhandlung beschrieben. Für statisch unbestimmte bolzenverbundene Rahmen, zusammengesetzt aus Materialien welche eine Spannungs-Beanspruchungsbeziehung in in der Form $\varepsilon = \varepsilon_0(\sigma/\sigma_0)^n$ haben und für fest verbundene Rahmen, zusammengesetzt aus Gliedern welche Moment-Biegungsbeziehung in der Form $\phi = \phi_0(M/M_0)^n$ haben, wird gezeigt, das keine Wiederverteilung von Wirkungen vorkommt, als die Belastung zunimmt. Die Anwendbarkeit der Methode von "Gemischten Systemen" wird vorgeführt. Für Moment-Biegungsbeziehungen welche eine bestimmt Neigung bei M = null, haben, ist in einer Anzahl von besonderen Fällen gezeigt, das die Wiederverteilung bei zunehmender Ladung nur gering ist und das die Verteilung von Wirkungen denen ähnlich ist, die mit einer linearen Analyse erhalten werden.

Абстракт—В этой статье описывается общий метод анализа каркасной конструкции компонированной на нелинейном эластическом материале. Для статически неопределимых закрепленных шарнирами ферм, составленных из материала, имеющего отношение напряжение-деформация формы $\epsilon = \epsilon_0(\sigma/\sigma_0)^n$ и для жёстких рам, составленных из элементов, имеющих отношения момент-кривизна формы $\phi = \phi_0(M/M_0)^n$, показано, что при увеличении нагрузки не происходит перераспределения действия. Демонстрируется применимость метода "смешанных систем". Для отношений момент-кривизна, имеющих определённый наклон при $M = 0$ в некотором числе особенных случаев показывается, что перераспределение под увеличивающейся нагрузкой только малое и, что распределение действия подобно распределению, получаемому при линейных анализах.